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Bianchi I Cosmological Model and the No-Boundary Condition

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A study of the Bianchi I cosmological model is done from both the classical and quantum mechanical points of view. The field equations and their solutions are discussed in the classically forbidden region and classical region. Also the no-boundary wave function is evaluated using the concept of microsuperspace and the Hawking–Hartle proposal.

1. INTRODUCTION

This paper deals with the study of classical and quantum cosmological phenomena for the spatially homogeneous anisotropic Bianchi I space-time. The Einstein field equations are solved in both Euclidean and Lorentzian regions with boundary conditions based upon the no-boundary proposal of Hartle and Hawking. In addition, the Hamilton–Jacobi function is constructed from the Euclidean action, which is the solution of the Euclidean field equations, by analytic continuation. These are discussed in Section 2. The wave function for the no-boundary proposal is evaluated in Section 3 with the concept of microsuperspace. The path integral reduces to a single integration and is evaluated by the method of steepest descent. Different wave functions for different choices of the contour are found, and are given in tabular form. Therefore, the HH proposal does not lead to a unique wave function.

2. BIANCHI I COSMOLOGICAL MODEL

The spatially homogeneous metric ansatz in the Bianchi type I model is (Louko, 1988)

$$dS^2 = \rho^2[-N^2(t)\, dt^2 + a^2(t)\, dx^2 + b^2(t)\, dy^2 + c^2(t)\, dz^2] \tag{2.1}$$

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The overall prefactor is $\rho^2 = G/2\pi^2$. The coordinates x, y, and z are dimensionless and are periodic with period 2π. The surfaces of constant t have the topology of a three-torus. The minisuperspace variables a, b, c, and N are dimensionless quantities.

In this section, we consider a massive scalar field $\phi(t)$ in the above model. The corresponding action is (Louko, 1988)

$$I = -\tfrac{1}{2}\int Nabc\left[\frac{1}{N^2}\left(\frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} + \frac{\dot a}{a}\frac{\dot b}{b} - \dot\phi^2\right) + m^2\phi^2\right] dt$$

$$= \int L\, dt \tag{2.2}$$

According to Hartle and Hawking (1983), due to inflation at early stages of evolution of the universe, the effective cosmological constant has to decay and finally vanish with the evolution of the universe, as our universe is not expanding exponentially today (Chakraborty, 1990*a*). Here the scalar field ϕ is assumed initially very large and almost constant, i.e., $\phi \simeq \text{const} = \phi_0$ at the very early stages of the evolution and the mass term is taken as the effective cosmological constant (Esposito and Platania, 1988). This region is known as Euclidean or classically forbidden. The field equations in this region are

$$\begin{aligned}
&\frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot b}{b}\frac{\dot c}{c} + N^2\lambda = 0\\
&\frac{\ddot c}{c} + \frac{\ddot a}{a} + \frac{\dot c}{c}\frac{\dot a}{a} + N^2\lambda = 0\\
&\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\frac{\dot b}{b} + N^2\lambda = 0
\end{aligned} \tag{2.3}$$

and the constraint equation is

$$\frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} + N^2\lambda = 0$$

Here derivatives are with respect to Euclidean time τ and $\lambda = m^2\phi_0^2$. The solution of these field equations with the boundary conditions (Louko, 1988)

$$c(0) = 0, \qquad \dot c(0) = 1, \qquad \dot a(0) = \dot b(0) = 0 \tag{2.4}$$

[(0) stands for $\tau=0$] are

$$a(\tau)=K_1\cos^{2/3}\left(\frac{(3\lambda)^{1/2}N\tau}{2}\right)$$
$$b(\tau)=K_2\cos^{2/3}\left(\frac{(3\lambda)^{1/2}N\tau}{2}\right) \tag{2.5}$$
$$c(\tau)=\frac{2}{(3\lambda)^{1/2}}\sin\left(\frac{(3\lambda)^{1/2}N\tau}{2}\right)\sec^{1/3}\left(\frac{(3\lambda)^{1/2}N\tau}{2}\right)$$

where K_1, K_2 are arbitrary constants.

Now, according to the HH proposal, the ground-state wave function is a Euclidean functional integral taken over compact four-metrics and regular matter fields. So in the semiclassical approximation the wave function reduces to (Chakraborty, 1990*a*)

$$\psi=c\exp(-I_e) \tag{2.6}$$

where I_e is the Euclidean action of the above solutions of the Euclidean field equations and is given by

$$I_E=\frac{abc\sqrt{\lambda}}{4\sqrt{3}}\left(T-\frac{3}{T}\right) \tag{2.7}$$

with

$$T=2^{1/3}\Bigg/\left\{\left[1+\left(1+\frac{c^6\lambda^3}{16}\right)^{1/2}\right]^{1/3}-\left[\left(1+\frac{c^6\lambda^3}{16}\right)^{1/2}-1\right]^{1/3}\right\}$$

In the classical region the wave function is oscillatory in nature and the WKB ansatz gives (Chakraborty, 1990*a*; Fang and Ruffin, 1987)

$$\psi=\mathrm{Re}[c\exp(iS)] \tag{2.8}$$

where we assume

$$\frac{\nabla^2 C}{C}\ll(\nabla S)^2$$

So the prefactor varies slowly compared to the Hamilton–Jacobi function. The gradient of the HJ function gives the direction of the classical trajectories in superspace (Chakraborty, 1990*a*). The analytic continuation of the Euclidean action is the choice of HJ function based on the Hartle–Hawking proposal and we have

$$S \simeq \frac{abc\sqrt{\lambda}}{4\sqrt{3}}\left(T - \frac{3}{T}\right), \qquad \lambda = m^2\phi^2 \tag{2.9}$$

The solutions of the Lorentzian field equations are also obtained from (2.5) by rotating the time axis as

$$\begin{aligned} a(t) &= K_1 \sinh^{2/3}(\delta t) \\ b(t) &= K_2 \sinh^{2/3}(\delta t) \\ c(t) &= [2/(3\lambda)^{1/2}] \cosh(\delta t) \operatorname{cosech}^{1/3}(\delta t) \\ \phi(t) &= \phi_0 \end{aligned} \tag{2.10}$$

with $\delta = (3\lambda)^{1/2} N/2$.

Now, the first integral of the system is (Esposito and Platania, 1988)

$$p_a = \frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}}, \qquad p_b = \frac{\partial S}{\partial b} = \frac{\partial L}{\partial \dot{b}}$$

$$p_c = \frac{\partial S}{\partial c} = \frac{\partial L}{\partial \dot{c}}, \qquad p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}$$

where L is given by (2.2). These first-order differential equations are coupled and are complicated in form, so they cannot be solved exactly. Therefore no conclusion regarding Lorentzian trajectories is possible.

3. NO-BOUNDARY WAVE FUNCTION USING MICROSUPERSPACE MODEL

The concept of microsuperspace was formed by Halliwell and Louko (1989, 1990) using the recent study of Regge calculus (Regge, 1961) by Hartle (1989). In these models one is directly concerned with four-geometries, while minisuperspace models involve only three-geometries.

The wave function of the universe for the no-boundary proposal of Hartle and Hawking (1983) is given by the path integral

$$\psi(h_{ij}) = \int_\Gamma D(g_{\mu\gamma}) \exp[-I_E(g_{\mu\gamma})] \tag{3.1}$$

where I_E is the Euclidean action of the gravitational field with a cosmological constant. In this section the path integral will be evaluated using this concept of microsuperspace for the metric ansatz in the Bianchi I model. The class of Euclidean four-metric is taken to obey the Bianchi I ansatz

$$ds^2 = \rho^2[N^2(\tau)\,d\tau^2 + a^2(\tau)\,dx^2 + b^2(\tau)\,dy^2 + c^2(\tau)\,dz^2] \qquad (3.2)$$

The Euclidean version of the Einstein–Hilbert action with a positive cosmological term for this metric is (Louko, 1988)

$$I = \int_0^{\tau'} L\,d\tau - \left[\frac{1}{2N}\frac{d}{d\tau}(abc)\right]_{\tau=0} \qquad (3.3)$$

where the Lagrangian has the expression

$$L = \frac{1}{2}\left[-\frac{1}{N}(a\dot{b}\dot{c} + b\dot{c}\dot{a} + c\dot{a}\dot{b}) + N\lambda abc\right] \qquad (3.4)$$

$(\cdot \equiv d/d\tau)$.

As in the path integral (3.1), Γ corresponds to a class of microsuperspace four-metric labeled by an arbitrary parameter (say r) (Chakraborty, 1990*b*), so we consider the class of four-metrics (3.2) for which the scale factors have the expressions [see equation (2.5)]

$$\begin{aligned} a(\tau) &= K_1 \cos^{2/3}(N\tau/r) \\ b(\tau) &= K_2 \cos^{2/3}(N\tau/r) \\ c(\tau) &= r \sin(N\tau/r) \sec^{1/3}(N\tau/r) \end{aligned} \qquad (3.5)$$

Inserting (3.5) into (3.3) and integrating, one finds that the action for this class of metric is

$$\begin{aligned} I(r, \alpha, \beta, \gamma) = \frac{\alpha\beta}{12}[&-2\cos^{-4/3}(N\tau^*/r) - 4\cos^{2/3}(N\tau^*/r) \\ &+ 3\lambda r^2 \cos^{-2/3}(N\tau^*/r)] \end{aligned} \qquad (3.6)$$

where

$$\alpha = a(\tau^*), \qquad \beta = b(\tau^*), \qquad \gamma = c(\tau^*) \qquad (3.7)$$

Let us define

$$\mu = \cos^{2/3}(N\tau^*/r) \qquad (3.8)$$

Then I in equation (3.6) simplifies to

$$I(\mu, \alpha, \beta, \gamma) = (\alpha\beta/12)(-2/\mu^2 - 4u + 3\lambda\gamma^2/\mu) \qquad (3.9)$$

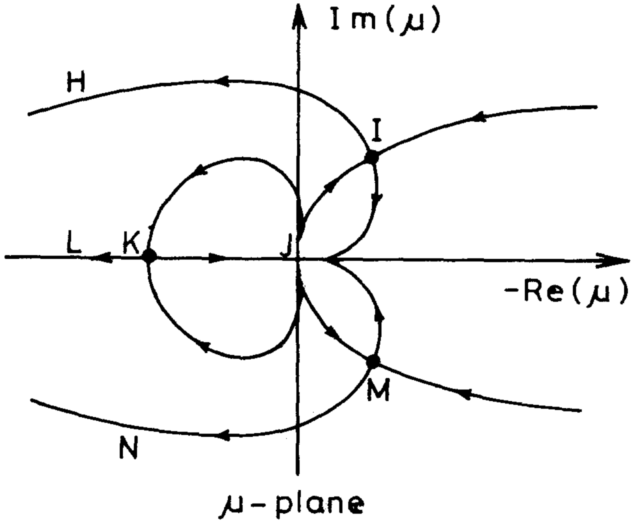

Fig. 1. The steepest-descent paths in the complex μ-plane for positive cosmological constant. In this and the following figures, the arrows point downhill, and there is an essential singularity at $\mu=0$.

Hence, according to the Hartle–Hawking proposal, the wave function given by the path integral now reduces to a single ordinary integration over μ (Chakraborty, 1990*b*)

$$\psi(\alpha, \beta, \gamma) = \int_\Gamma d\mu \; v(\mu, \alpha, \beta, \gamma) \exp[-I(\mu, \alpha, \beta, \gamma)] \tag{3.10}$$

where the contour Γ in the complex μ-plane is such that (3.10) converges and v is a measure of integration. According to Halliwell and Louko (1989, 1990), this measure may only affect the prefactor to some extent.

The integration in equation (3.10) is now evaluated (with measure $v=1$) by the saddle point method. So the path Γ now corresponds to

Table I. The Wave Function Based on the HH Proposal in the Leading-Order Saddle-Point Approximation

Contour	Wave function
HIJ	$\exp(-I_2)\exp(-iI_3)$
NMJ	$\exp(-I_2)\exp(+iI_3)$
HIJMN	$\exp(-I_2)\cos(-I_3+\pi/4)$
LKJ	$\exp(-I_1)$

steepest-descent contours. The saddle points of the contours are the roots of the cubic equation

$$4\mu^3 + 3\lambda\gamma^2\mu - 4 = 0 \qquad (3.11)$$

So we have one real and two complex saddle points given by

$$\begin{aligned} \mu_1 &= (t+\tfrac{1}{2})^{1/3} - (t-\tfrac{1}{2})^{1/3} \\ \mu_2 &= -\tfrac{1}{2}\mu_1 + i\sqrt{\tfrac{3}{2}}[(t-\tfrac{1}{2})^{1/3} + (t+\tfrac{1}{2})^{1/3}] \\ \mu_3 &= -\tfrac{1}{2}\mu_1 - i\sqrt{\tfrac{3}{2}}[(t-\tfrac{1}{2})^{1/3} + (t+\tfrac{1}{2})^{1/3}] \end{aligned} \qquad (3.12)$$

with $t = (1 + \lambda^3\gamma^6/16)^{1/2}/2$.

Also note that the integrand has an essential singularity at $\mu = 0$. Let I_1, $I_2 \pm iI_3$ be the values of the action at $\mu = \mu_1$ and μ_2, μ_3, respectively, i.e.,

$$I_1 = I|_{\mu=\mu_1}, \qquad I_2 + iI_3 = I|_{\mu=\mu_2}, \qquad I_2 - iI_3 = I|_{\mu=\mu_3}$$

Now the steepest-descent paths are given by $I_m(I) = \text{const} = I$ (at the saddle points) and are shown in Figure 1. Table I shows the expression for the wave function for different convergent contours.

One may note that so far we have assumed that λ is real and positive. Now if λ is taken to be negative ($\lambda = -\theta^2$), then one can obtain the same results as before provided $\theta^2\gamma^2 < (16)^{1/3}$ and $t = (1 - \theta^6\gamma^6/16)^{1/2}/2$.

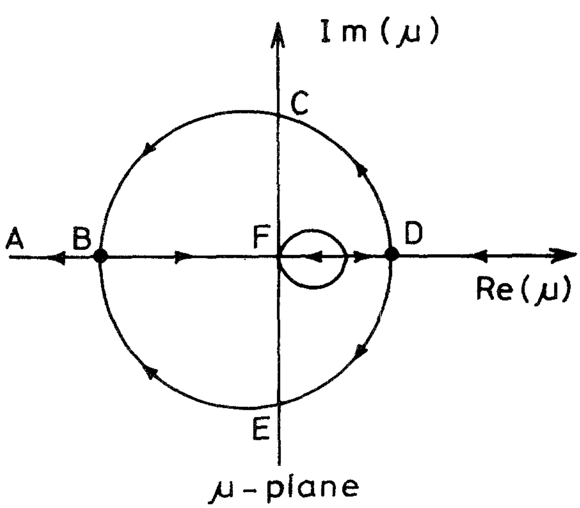

Fig. 2. The steepest-descent paths with negative cosmological constant ($= -\theta^2$) and $\theta^2\gamma^2 > (16)^{1/3}$.

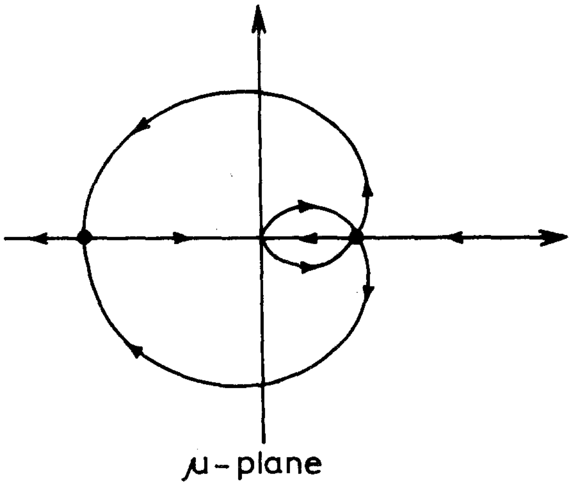

Fig. 3. The steepest-descent paths in the limiting case $\theta^2 r^2 = (16)^{1/3}$.

For $\theta^2\gamma^2 \geq (16)^{1/3}$, we have only real saddle points. In fact, for $\theta^2\gamma^2 > (16)^{1/3}$ there are three real saddle points given by

$$2\theta\gamma \cos\frac{\bar{\theta}}{3}, \qquad 2\theta\gamma \cos\frac{\bar{\theta}+2\pi}{3}, \qquad 2\theta\gamma \cos\frac{\bar{\theta}+4\pi}{3}$$

with

$$\sec\bar{\theta} = (\theta, \gamma)^3$$

The convergent contours through these saddle points are given in Figure 2. The limiting case $\theta\gamma = 4^{1/3}$ is shown in Figure 3.

Thus, we obtain a similar conclusion to Halliwell and Louko (1989, 1990) that the wave function based on the Hartle–Hawking proposal is not unique as it does not fix the contour.

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